Seminar Algebra, Geometry, Topology and Applications

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Introduction to Novikov algebras

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Overview

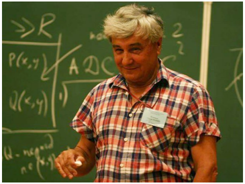
These algebras were used to study problems connected with physics both by

- A. A. Belinskii and S. P. Novikov[1].
- I. M. Gelfand and I. Ya. Dorfman[2].

The term "Novikov algebra" was given by J. M. Osborn.

[1] Poisson brackets of hydrodynamic type, Frobenius algebras and Lie algebras, Dokl. Akad. Nauk SSSR 283 (1985), 1036-1039; English transl. Soviet Math. Dokl. 32 (1985), 228-231.

[2] Hamiltonian operators and algebraic structures related to them, Funct. Anal. Appl. 13 (1980) 248-262.



- Sergue Petrovitch Novikov was born on 20 March 1938 in Gorky, Soviet Union (now Nizhny Novgorod, Russia).
- 1965 Doctor of Science in Physics and Mathematics. Thesis title: "Homotopy equivalent smooth manifolds". (Prof. M. M. Postnikov – adviser).
- 1970 Fields Medal.
- 2005 Wolf Prize in Mathematics.
- 1985-1996 President of the Moscow Mathematical Society.
- 1983–1986 and 2000–2002 Member of Fields Medal Committees of The International Mathematical Union.
- 2008-2010 Member of the Abel Prize Committee.

Definition

A Novikov algebra A is a vector space over a field K with a bilinear product $(x, y) \to xy$ satisfying

$$(x_1, x_2, x_3) = (x_2, x_1, x_3)$$

and

$$(x_1 x_2)\, x_3 = (x_1 x_3)\, x_2$$

for $x_1, x_2, x_3 \in A$, where

$$(x_1, x_2, x_3) = (x_1 x_2)\, x_3 - x_1\,(x_2 x_3)$$

Novikov algebras are a special class of left-symmetric algebras which only satisfy the first equation. Left-symmetric algebras are non-associative algebras arising from the study of affine manifolds, affine structures... The beauty of a Novikov algebra is that the left multiplication operators form a Lie algebra and the right multiplication operators are commutative.

Examples in low dimensions[3]

There are two Novikov algebras in dimension one: the complex field $C = \{\mathbb{C}e / ee = e\}$ and the one-dimensional trivial Novikov algebra $(\mathrm{T} = \{\mathbb{C}e / ee = 0\}$.

Characteristic matrix	Associativity	Lie algebra $\mathcal{G}(A)$
(T1) $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	Associative	Abelian
(T2) $\begin{pmatrix} e_2 & 0 \\ 0 & 0 \end{pmatrix}$	Associative	Abelian
(T3) $\begin{pmatrix} 0 & 0 \\ -e_1 & 0 \end{pmatrix}$	Non-associative	$[e_1, e_2] = e_1$
(N1) $\begin{pmatrix} e_1 & 0 \\ 0 & e_2 \end{pmatrix}$	Associative	Abelian
(N2) $\begin{pmatrix} e_1 & 0 \\ 0 & 0 \end{pmatrix}$	Associative	Abelian
(N3) $\begin{pmatrix} e_1 & e_2 \\ e_2 & 0 \end{pmatrix}$	Associative	Abelian
(N4) $\begin{pmatrix} 0 & e_1 \\ 0 & e_2 \end{pmatrix}$	Associative	Isomorphic to (T3)
(N5) $\begin{pmatrix} 0 & e_1 \\ 0 & e_1 + e_2 \end{pmatrix}$	Non-associative	Isomorphic to (T3)
(N6) $\begin{pmatrix} 0 & e_1 \\ le_1 & e_2 \end{pmatrix}$ $l \neq 0, 1$	Non-associative	Isomorphic to (T3)

[3] Chengming Bai and Daoji Meng, The classification of Novikov algebras in low dimensions, J. Phys. A: Math. Gen. 34 (2001) 1581-1594.

Basic facts

Lemma

Let $(A, \cdot)$ be a Novikov algebra. Then we have, for all $x, y, z \in A$

$$[x, y] \cdot z + [y, z] \cdot x + [z, x] \cdot y = 0$$
$$x \cdot [y, z] + y \cdot [z, x] + z \cdot [x, y] = 0$$

Proof.

1. $$\begin{aligned}[x, y] \cdot z + [y, z] \cdot x + [z, x] \cdot y &= (x \cdot y) \cdot z - (y \cdot x) \cdot z + (y \cdot z) \cdot x \\ &\quad -(z \cdot y) \cdot x + (z \cdot x) \cdot y - (x \cdot z) \cdot y \\ &= 0\end{aligned}$$

2. $$\begin{aligned}x \cdot [y, z] + y \cdot [z, x] + z \cdot [x, y] &= [y, z] \cdot x + [x, [y, z]] + [z, x] \cdot y \\ &\quad + [y, [z, x]] + [x, y] \cdot z + [z, [x, y]] \\ &= 0\end{aligned}$$

$\square$

Proposition

Let $(A, \cdot)$ be a Novikov algebra and I, J be two-sided ideals of A. Then $I \cdot J$ is also a two-sided ideal of A.

Proof.

Let $a \in A, x \in I$ and $y \in J$. Then the identity
$a \cdot (x \cdot y) = (a \cdot x) \cdot y + x \cdot (a \cdot y) - (x \cdot a) \cdot y$ shows that $a \cdot (x \cdot y) \in I \cdot J$.
Because of $(x \cdot y) \cdot a = (x \cdot a) \cdot y$ we also have $(x \cdot y) \cdot a \in I \cdot J$ □

Proposition

Let $(A, \cdot)$ be a Novikov algebra and assume that I, J are two-sided ideals of A. Then $[I, J]$ is also a two-sided ideal of A.

Proof.

We have

$$0 = [L(x) - \mathrm{ad}(x), L(y) - \mathrm{ad}(y)] = L([x, y]) + \mathrm{ad}([x, y]) - [\mathrm{ad}(x), L(y)] - [L(x), \mathrm{ad}(y)]$$

Let $a \in A, x \in I$ and $y \in J$. Thus

$$\begin{aligned} 0 =& [x, y] \cdot a + [[x, y], a] - [x, y \cdot a] + y \cdot [x, a] - x \cdot [y, a] + [y, x \cdot a] \\ =& [x, y] \cdot a + [[x, y], a] - [x, y \cdot a] + [y, x \cdot a] + a \cdot [x, y] \\ =& [x, y] \cdot a + [[x, y], a] - [x, y \cdot a] + [y, x \cdot a] + [x, y] \cdot a + [a, [x, y]] \\ =& 2[x, y] \cdot a + [y, x \cdot a] - [x, y \cdot a] \end{aligned}$$

From this we deduce

$$\begin{aligned} &[x, y] \cdot a = \tfrac{1}{2}([x, y \cdot a] - [y, x \cdot a]) \in [I, J], \\ &a \cdot [x, y] = [x, y] \cdot a + [a, [x, y]] \in [I, J] \end{aligned}$$

which was to be shown. □

Let $(A,\cdot)$ be a Novikov algebra and $\mathfrak{g}_A$ its underlying Lie algebra. Denote by

$$\begin{array}{l} \mathfrak{g}_A^0 = \mathfrak{g}_A \\ \mathfrak{g}_A^{i+1} = \left[\mathfrak{g}_A, \mathfrak{g}_A^i\right] \end{array}$$

the terms of the lower central series of $\mathfrak{g}_A$. Furthermore denote by

$$\mathfrak{g}_A^{(0)} = \mathfrak{g}_A$$

$\mathfrak{g}_A^{(i+1)} = \left[\mathfrak{g}_A^{(i)}, \mathfrak{g}_A^{(i)}\right]$ the terms of the derived series of g_A. Then the above proposition immediately implies the following result:

Corollary

Let $(A,\cdot)$ be a Novikov algebra. Then all $\mathfrak{g}_A^i$, and all $\mathfrak{g}_A^{(i)}$ are two-sided ideals of A

Denote the center of a Novikov algebra A by $Z(A) = \{x \in A \mid x \cdot y = y \cdot x$ for all $y \in A\}$. Note that $Z(A)$ is also the center of the associated Lie algebra $\mathfrak{g}_A$.

Proposition

Let $(A, \cdot)$ be a Novikov algebra. Then $Z(A)$ is a two-sided ideal of A and $Z(A) \cdot [A, A] = [A, A] \cdot Z(A) = 0$.

Proof.

Let $z \in Z(A)$. For any $b \in A$ we have the following two identities

$$[L(b), L(z)] = L([b, z]) = 0$$

$[R(b), R(z)] = 0$ Because $z \in Z(A)$ we have $R(z) = L(z)$. Hence we also have $[L(b), R(z)] = 0$, so that

$$0 = [L(b) - R(b), R(z)] = [\mathrm{ad}(b), R(z)]$$

In particular it follows $[b, a \cdot z] - [b, a] \cdot z = 0$ for all $a \in A$. By the previous Lemma we have $[b, a] \cdot z = 0$, hence $[b, a \cdot z] = 0$. Because this is true for every $b \in A$, we can conclude that $a \cdot z \in Z(A)$ Since $z \in Z(A)$ we also have $z \cdot a \in Z(A)$. □

Denote by $(x, y, z) = x \cdot (y \cdot z) - (x \cdot y) \cdot z$ the associator of three elements in A.

Proposition

Let A be a Novikov algebra and one of the elements x, y, z in $Z(A)$. Then $(x, y, z) = 0$

Proof.

In any LSA we have the identity

$$\begin{aligned} (x,y,z) &= (x,y,z) - (x,z,y) + (z,x,y) \\ &= x(yz) - (xy)z - x(zy) + (xz)y + z(xy) - (zx)y \\ &= x \cdot [y,z] + [z, x \cdot y] + [x,z] \cdot y. \end{aligned}$$

If $z \in Z(A)$, then this implies $(x, y, z) = 0$. If $y \in Z(A)$ then also $x \cdot y \in Z(A)$. Hence the above identity implies $(x, y, z) = 0$. The same argument shows the claim for $x \in Z(A)$. □

On the (non) existence of Novikov structures on Lie algebras

Conjecture (J. Milnor (1977), Advances in Mathematics.)

Every solvable Lie algebra admits a left-symmetric structure.

Counter example (Besnoit Y. (1995). J. of Differential Geometry)

The nilpotente Lie algebra of dimension 11 defined by the Lie brackets

$$
\begin{aligned}
[X_1, X_i] &= X_{i+1} \ \ , i = 2, \dots, 10,\ [X_2, X_4] = X_6,\ [X_2, X_8] = \frac{26}{5}X_{10} + \frac{28}{25}X_{11}, \\
[X_3, X_4] &= 3X_7 - X_8 - tX_9,\ [X_3, X_6] = -\frac{12}{5}X_9 - \frac{1}{25}X_{10} + \frac{1525t - 448}{2000}X_{11}, \\
[X_3, X_8] &= \frac{321}{80}X_{11},\ [X_4, X_6] = \frac{27}{5}X_{10} - \frac{24}{25}X_{11},\ [X_5, X_6] = \frac{1377}{80}X_{11},\ [X_2, X_3] = X_5, \\
[X_2, X_5] &= -2X_7 + X_8 + tX_9,\ [X_2, X_7] = -\frac{13}{5}X_9 + \frac{51}{25}X_{10} + \frac{2475t + 448}{2000}X_{11}, \\
[X_2, X_9] &= \frac{19}{16}X_{11},\ [X_3, X_5] = 3X_8 - X_9 - tX_{10},\ [X_3, X_7] = -\frac{39}{5}X_{10} + \frac{23}{25}X_{11}, \\
[X_4, X_5] &= \frac{27}{5}X_9 - \frac{24}{25}X_{10} + \frac{448 - 3525t}{2000}X_{11},\ [X_4, X_7] = -\frac{189}{16}X_{11},\ \ t \in \mathbb{R}.
\end{aligned}
$$

admits no left-symmetric structure.

Proposition

Every 2-step nilpotent Lie algebra admits a Novikov structure.

Proof.

Let $\mathfrak{g}$ be a 2-step nilpotent Lie algebra. We put for any $x, y \in \mathfrak{g}$

$$x \cdot y = \frac{1}{2}[x, y].$$

This product is compatible with the Lie brackets, associative and verifies

$$[R_x, R_y] = 0.$$

$\square$

Theorem (J. Scheuneman (1974), Proc. Amer. Math. Soc.)

Any three-step nilpotent Lie algebra admits a left-symmetric structure.

Proof of Scheueman's result

Let f be an endomorphism of the underlying vector space of a Lie algebra $\mathfrak{g}$ such that

$$f([x,y]) - [f(x), f(y)] \in \mathcal{Z}(\mathfrak{g})$$

for all $x, y \in \mathfrak{g}$. Such an endomorphism is called a q-homomorphism of Lie algebras. An endomorphism d of the linear space $\mathfrak{g}$ is called an f derivation if

$$d[x,y] = [dx, fy] + [fx, dy]$$

for all $x, y \in \mathcal{G}$. In particular a derivation is an Id derivation.

Proposition

Let $\mathfrak{g}$ be a Lie algebra which admits an invertible f-derivation d. Then the product defined by

$$x \cdot y = d^{-1}\left[f(x), d(y)\right]$$

is left-symmetric.

Proposition

Every 3 step nilpotent Lie algebra has an invertible f derivation d.

Proof.

Every 3 step nilpotent can be decomposed as

$$\mathcal{G} = \mathcal{G}_0 \oplus \mathcal{G}_1 \oplus \mathcal{G}_2$$

where $\mathcal{G}_0 = [\mathcal{G}, [\mathcal{G}, \mathcal{G}]] \subset \mathcal{Z}(\mathcal{G}), \mathcal{G}_1$ is a supplement of $\mathcal{G}_0$ in $[\mathcal{G}, \mathcal{G}]$ and $\mathcal{G}_2$ is a supplement of $[\mathcal{G}, \mathcal{G}]$ in $\mathcal{G}$. Define

$$f(x) = a_i x \text{ for } x \in \mathcal{G}_i$$

where $a_1 = 4/9, a_2 = 2/3$, and

$$d(x) = \alpha_i x \text{ for } x \in \mathcal{G}_i$$

with $\alpha_0 = \alpha_1 = \frac{4}{3}\alpha_2 \neq 0.$ One can shows that d is an invertible f derivation. □

Proposition (D.Burde-K. Dekimpe-K. Vercammen (2008), Linear Algebra and its Applications.)

Consider the following three-step nilpotent Lie algebra with basis $(x_1, \dots, x_{13})$ *and non-trivial Lie brackets*

$$\begin{array}{ll}
[x_1, x_2] = x_5, & [x_3, x_4] = -x_5 \\
[x_1, x_4] = x_6, & [x_3, x_5] = -x_{11} \\
[x_1, x_6] = x_{10}, & [x_3, x_8] = x_9 \\
[x_1, x_7] = x_{11}, & [x_4, x_5] = -x_{12} \\
[x_1, x_8] = x_{12}, & [x_4, x_6] = x_9 \\
[x_2, x_3] = x_7, & [x_4, x_7] = x_9 + x_{13}. \\
[x_2, x_4] = x_8, & \\
[x_2, x_5] = x_{13}, & \\
[x_2, x_7] = x_{13}, &
\end{array}$$

This Lie algebra does not admit a Novikov structure.

One of the most fundamental examples for solvable, resp. nilpotent Lie algebras are the Lie algebras of upper-triangular, resp. strictly upper triangular matrices of size n over a field k, which we denote by $\mathfrak{t}(n, k)$, resp. $\mathfrak{n}(n, k)$. It is therefore natural to ask, which of those Lie algebras admit a Novikov structure. It turns out that such structures exist only in very small dimensions.

Proposition (D.Burde-K. Dekimpe-K. Vercammen (2008), Linear Algebra and its Applications.)

- *The Lie algebra $\mathfrak{n}(n, k)$ admits a Novikov structure if and only if $n \leqslant 4$.*
- *The Lie algebra $\mathfrak{t}(n, k)$ admits a Novikov structure if and only if $n \leqslant 2$.*

Proposition (D.Burde-K. Dekimpe (2006), Journal of Geometry and Physics.)

Any filiform Lie algebra with abelian commutator algebra admits a Novikov structure.

Simple Novikov algebras

Let A be an algebra over K and assume that the product is non-trivial.

Definition

The algebra A is called simple if every two-sided ideal in A is equal to A or equal to 0 .

Question (S. P. Novikov)

Whether there exist simple Novikov algebras?

Simple LSAs

The classification of simple LSAs is only known in low dimensions. Up to LSA-isomorphism there is only one 2 -dimensional simple complex LSA. It is given by $A = \mathbb{C}x \oplus \mathbb{C}y$ with product

$$x \cdot x = 2x, x.y = y, y.x = 0, y.y = x$$

Theorem (D. Burde (1998), Manuscripta Mathematica 95 (1998))

Let A be a simple 3 -dimensional LSA over $\mathbb{C}$. Then its Lie algebra $\mathfrak{g}$ is isomorphic to $\mathfrak{r}_{3,\lambda} = \langle e_1, e_2, e_3 | \, [e_1, e_2] = e_2, [e_1, e_3] = \lambda e_3 >$ with $|\lambda| \leq 1, \lambda \neq 0$ and A is isomorphic to exactly one of the following algebras $A_{1,\lambda}$ and A_2

$$\begin{array}{lll} e_1 \cdot e_1 = (\lambda + 1)e_1 & e_1 \cdot e_3 = \lambda e_3 & e_3 \cdot e_2 = e_1 \\ e_1 \cdot e_2 = e_2 & e_2 \cdot e_3 = e_1 & \end{array}$$

and

$$\begin{array}{llll} e_1 \cdot e_1 & = \frac{3}{2}e_1 & e_1 \cdot e_3 = \frac{1}{2}e_3 & e_3 \cdot e_2 = e_1 \\ e_1 \cdot e_2 & = e_2 & e_2 \cdot e_3 = e_1 & e_3 \cdot e_3 = -e_2 \end{array}$$

Theorem (D. Burde (1996), J. Algebra-O. Baues (1999), Trans. Amer. Math. Soc.)

There exist infinitely many non-isomorphic LSA-structures on $\mathfrak{gl}(n, K)$. They are simple as LSAs.

Proposition

Let A be an LSA with reductive Lie algebra of 1 -dimensional center. Then A is simple.

Proof.

Let $\mathfrak{g} = \mathfrak{g}_A = \mathfrak{s} \oplus \mathfrak{z}$ be the Lie algebra with center $\mathfrak{z} \simeq K$. Suppose I is a proper two-sided ideal in A. Then it is also a proper Lie ideal in $\mathfrak{g}$ and both I and $\mathfrak{g}/I$ inherit an LSA-structure from A. Since a semisimple Lie algebra does not admit any LSA-structures, we conclude that I must be equal to $\mathfrak{s}_1 \oplus K$, where $\mathfrak{s}_1$ is a semisimple ideal of $\mathfrak{s}$. Hence $\mathfrak{g}/I$ is semisimple and admits an LSA-structure. This is a contradiction. $\square$

Simple Novikov algebras

Question (S. P. Novikov)

Whether there exist simple Novikov algebras?

Theorem (E. Zelmanov (1987), Soviet Math. Dokl.)

A simple Novikov algebra A over an algebraically closed field $\mathbb{K}$ with characteristic zero is isomorphic to $\mathbb{K}$.

Osborn and Xu gave a complete classification of finite-dimensional simple Novikov algebras over an algebraically closed field with prime characteristic. They also found several classes of infinite-dimensional simple Novikov algebras [J. M. Osborn (1992), Nova J. Algebra Geom.-X. Xu (1997), J. Algebra].

Example (A two-dimensional simple Novikov algebra over $\mathbb{R}$)

Over the field $\mathbb{F} = \mathbb{R}$, let us consider the two-dimensional commutative associative algebra $A_{2,\mathbb{R}}$ defined by the following multiplication table:
$e_1 \cdot e_1 = e_1, \quad e_1 \cdot e_2 = e_2 \cdot e_1 = e_2, \quad e_2 \cdot e_2 = -e_1$. Being commutative, $A_{2,\mathbb{R}}$ is a Novikov algebra; We can easily see that $A_{2,\mathbb{R}}$ is simple.

Theorem (M. Guediri (2016), Communications in Algebra)

Let A be simple Novikov algebra over $\mathbb{R}$. Then, A is isomorphic to either $A_{2,\mathbb{R}}$ or the field $\mathbb{R}$.

Alternative or flexible left-symmetric algebras

Definition

An alternative algebra A over $\mathbb{K}$ is an algebra in which

$$(x, x, y) = (y, x, x) = 0, \ \forall x, y \in A.$$

Proposition

Let A be a left-symmetric algebra which is also (left) alternative (char$(\mathbb{K}) \neq 2$). Then A is associative.

Proof.

$$\begin{aligned} 0 = (x + y, x + y, z) &= (x, x, z) + (x, y, z) + (y, x, z) + (y, y, z) \\ &= (x, y, z) + (y, x, z). \end{aligned}$$

Thus $(x, y, z) = -(y, x, z) = -(x, y, z)$ and hence A is associative. $\square$

Definition

An algebra A over $\mathbb{K}$ is flexible if

$$(x, y, x) = 0, \ \forall x, y \in A.$$

Proposition

Let A be a left-symmetric algebra which is also flexible ($char(\mathbb{K}) \neq 2$). Then A is associative.

Proof.

$$\begin{aligned} 0 = (x+z, y, x+z) &= (x, y, x) + (x, y, z) + (z, y, x) + (z, y, z) \\ &= (x, y, z) + (z, y, x). \end{aligned}$$

Thus

$$(x, y, z) = -(z, y, x) = -(y, z, x) = (x, z, y) = (z, x, y) = -(y, x, z) = -(x, y, z)$$

and hence A is associative. $\square$

Quadratic Novikov algebras

Definition

A quadratic Novikov algebra is a couple (A, f) *where* A *is a Novikov algebra and* f *is a non-degenerate symmetric bilinear form* $f : A \times A \to \mathbb{K}$ *such that*

$$f(xy, z) = f(x, yz), \quad \forall x, y, z \in A.$$

Definition

A quadratic Lie algebra is a couple (g, f) where g is a Lie algebra and f is a nondegenerate symmetric bilinear form satisfying

$$f([x,y],z) = f(x,[y,z]), \quad \forall x,y,z \in \mathrm{g}$$

Remark

Let (A, f) be a quadratic Novikov algebra and A_L be the sub-adjacent Lie algebra of A. Then (A_L, f) is a quadratic Lie algebra.

Indeed, since (A, f) is a quadratic Novikov algebra, then

$$\begin{aligned} f([x,y],z) &= f(xy,z) - f(yx,z) \\ &= f(x,yz) - f(y,xz) \\ &= f(x,yz) - f(x,zy) \\ &= f(x,[y,z]) \end{aligned}$$

for any $x, y, z \in A$. It follows that (A_L, f) is a quadratic Lie algebra.$\square$

Proposition

Let (A, f) be a quadratic Novikov algebra. Then A_L is 2-step nilpotent.

Proof.

Proof. For any $a, b, c, d \in A$,

$$\begin{aligned} f(a[b,c],d) = f(a,[b,c]d) &= f(a, b(cd) - c(bd)) \\ &= f((cd)a, b) - f((bd)a, c) \\ &= f((ca)d, b) - f((ba)d, c) \\ &= f(ca, db) - f(ba, dc) \\ &= f(b(ca), d) - f(c(ba), d) \\ &= f([b,c]a, d). \end{aligned}$$

It follows that $a[b,c] = [b,c]a$, and hence $[a,[b,c]] = 0$. $\square$

On the other hand, for a 2-step nilpotent quadratic Lie algebra (A, f), define a bilinear product on A by $xy = \frac{1}{2}[x, y]$ for any $x, y \in A$, then (A, f) is a quadratic Novikov algebra.

Thus, to get the classification of A_L it is enough to get the classification of 2-step nilpotent quadratic Lie algebras. Here a partial result:

Denote by $L_{6,26}$ the 6 -dimensional 2 -step nilpotent Lie algebra defined by the Lie brackets

$$[x_1, x_2] = x_4, [x_1, x_3] = x_5, [x_2, x_3] = x_6$$

Theorem (Benayadi-Lebzioui)

A 2-step Lie algebra $\mathfrak{g}$ admits a quadratic metric of signature $(3, n-3)$ where $n \geq 6$ if and only if $\mathfrak{g}$ is isomorphic to a trivial central extension of $\mathrm{L}_{6,26}$. Furtheremore, the restriction of the metric to $\mathrm{L}_{6,26}$ is non degenerate of signature (3,3) and defined by the only non vanishing scalar products $\langle x_1, x_6 \rangle = \langle x_2, x_5 \rangle = \langle x_3, x_4 \rangle = \alpha \neq 0$.

Proposition (M. BORDEMANN (1997), Acta Math. Univ. Comenianae)

Let A be a left-symmetric algebra over a field of characteristic different from 2 endowed with a quadratic form f. Then A is an associative algebra.

Proof.

Let $a, b, c, d \in A$. Then we have the following identity for the associator $(a, b, c) := (ab)c - a(bc)$

$$f((a, b, c), d) = f((ab)c - a(bc), d) = f(ab, cd) - f(a, (bc)d) = f(a, b(cd) - (bc)d)$$

hence

$$f((a, b, c), d) = -f((b, c, d), a) \forall a, b, c, d \in A$$

It follows that

$$\begin{aligned} f((a, b, c), d) &= -f((b, c, d), a) = -f((c, b, d), a) \\ &= +f((b, d, a), c) = +f((d, b, a), c) \\ &= -f((b, a, c), d) = -f((a, b, c), d) \end{aligned}$$

This clearly implies $2f((a, b, c), d) = 0$ for all $a, b, c, d \in A$, hence $(a, b, c) = 0$ by the nondegeneracy of f.$\square$

Double extension of quadratic associative algebras

Lemma

Let (A, f) be a quadratic associative algebra. If I is a two-sided ideal then $I^{\perp}$ is also a two-sided ideal and $I.I^{\perp} = I^{\perp}.I = 0$.

Proof.

Let $a \in A$, $b \in I$ and $x \in I^{\perp}$. We have

$$f(ax, b) = f(a, xb) = f(x, ba) = 0,$$

then $A.I^{\perp} \subset I^{\perp}$ and $I^{\perp}.I = 0$. From

$$f(xa, b) = f(x, ab) = 0 = f(a, bx),$$

we deduce that $I^{\perp}.A \subset I^{\perp}$ and $I.I^{\perp} = 0$. □

Double extension of quadratic associative algebras

Let $(B, \star, f_B)$ be a quadratic associative algebra. Let $A = \mathbb{K}e \oplus B \oplus \mathbb{K}d$ endowed by the form f defined by $f/_{B\times B} = f_B$, $f(e, B) = f(d, B) = 0$, e, d are isotropics and $f(e, d) = 1$. Let $x_0 \in B$ and $D \in End(B)$ such that

$$D \circ D^* = D^* \circ D,\ D(x_0) = D^*(x_0),\ D^2(a) = x_0 \star a,\ D(a \star b) = D(a) \star b,\ \forall a, b \in B.$$

We define on A the product

$$\begin{aligned} L_e &= R_e = 0, \\ a.b &= a \star b + f_B(D(a), b)e, \\ d.a &= D(a) + f_B(x_0, a)e, \\ a.d &= D^*(a) + f_B(x_0, a)e \\ d.d &= \lambda e + x_0. \end{aligned}$$

Then $(A, ., f)$ is a quadratic associative algebra. $(A, ., f)$ is called double extension of $(B, \star, f_B)$ according to D, x_0 and λ.

Double extension of quadratic associative algebras

Proposition

Let $(A, ., f)$ be a quadratic associative algebra of dimension n. If $Z(A)$ contains an isotropic (non nul) vector e such that $L_e = 0$ then $(A, ., f)$ is a double extension of a quadratic associative algebra $(B, \star, f_B)$ according to D, x_0 and λ.

Radicals of Novikov algebras

Radicals of left-symmetric algebras

For left-symmetric algebras, different types of radicals have been defined. We shall consider here three of them.
Recall that

Theorem (D. Segal (1992), Mathematische Annalen)

Let A be a finite-dimensional LSA over a field K of characteristic zero. Then the following conditions are equivalent:
(1) A is complete.
(2) A is right nil, i.e., $R(x)$ is a nilpotent linear transformation, for all $x \in A$.
(3) $R(x)$ has no eigenvalue in $K \backslash \{0\}$, for all $x \in A$.
(4) $\operatorname{tr}(R(x)) = 0$ for all $x \in A$.
(5) $Id + R(x)$ is bijective for all $x \in A$.

The (Koszul) radical of a left-symmetric algebra

Definition

Let A be an LSA and $I(A) = \{x \in A \mid \operatorname{tr} R(x) = 0\}$. The largest left ideal of A contained in $I(A)$ is called the radical of A and is denoted by $\mathrm{R}(A)$.

It turns out that $R(A)$ is nothing but the largest complete left ideal of A. This has been proved in [J. Helmstetter (1979). Ann. Inst. Fourier] for the case $\mathbb{F} = \mathbb{C}$, and in [Chang, K., Kim, H., Myung, H. C. (1999). Comm. Algebra] for $\mathbb{F} = \mathbb{R}$.
Note that A is complete if and only if $A = \mathrm{rad}(A)$. It is not clear whether this is a good definition of the radical of an LSA. Usually the radical should be a 2 -sided ideal in the algebra. Helmstetter in [J. Helmstetter (1979). Ann. Inst. Fourier] has constructed an LSA, B where $\mathrm{R}(B)$ is not a 2-sided ideal in general.

Let $(A, \cdot)$ be an LSA and set

$$B = \mathrm{End}(A) \oplus A$$

We may equipp this vector space with a left-symmetric product by

$$(f, a) \cdot (g, b) = (fg + [L(a), g], a \cdot b + f(b) + g(a))$$

for $a, b \in A$ and $f, g \in \mathrm{End}(A)$.

Proposition

The algebra B is an LSA. If A is not complete then $R(B) = 0$. If A is complete and the product in A is not identically zero then $\mathrm{R}(B)$ is not a 2 -sided ideal in A.

Example (Burde)

Define a 4-dimensional left-symmetric algebra A by the following product:

$$\begin{array}{llll} e_1 \cdot e_3 = e_3 & e_2 \cdot e_2 = 2e_2 & e_3 \cdot e_4 = e_2 & e_1 \cdot e_4 = -e_4 \\ e_2 \cdot e_3 = e_3 & e_4 \cdot e_3 = e_2 & e_2 \cdot e_4 = e_4 & \end{array}$$

and the other products equal to zero. Then $\mathrm{R}(A) = \mathrm{span}\,\{e_1\}$. This is not a right ideal in A.

The right multiplications are given by

$$R(e_1) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad R(e_2) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$
$$R(e_3) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad R(e_4) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \end{pmatrix}.$$

We see that $I(A) = \ker \operatorname{tr} R = \operatorname{span}\{e_1, e_3, e_4\}$. The largest left ideal in $I(A)$ is given by $\operatorname{span}\{e_1\}$. The solvable, non-nilpotent Lie algebra is given by

$$[e_1, e_3] = e_3, [e_2, e_3] = e_3$$
$$[e_1, e_4] = -e_4, [e_2, e_4] = e_4$$

The left (resp. right)-nilpotent radical

Let A be a left-symmetric algebra over a field $\mathbb{F}$ of characteristic zero, and I a two-sided ideal of A. We say that I is left-nilpotent if there exists some fixed integer $n \geq 1$ such that $L_{a_1} \cdots L_{a_n} = 0$ for all $a_i \in I$.

Lemma (K. S. Chang, H. Kim, H. Lee (1999). Commun. Algebra)

Let A be a finite-dimensional LSA. If I and J are left-nilpotent ideals of A, then so is $I + J$.

Thus every finite-dimensional left-symmetric algebra A has a unique maximal left nilpotent ideal $L(A)$, called the left radical of A, and

$$L(A) \subset R(A).$$

This is a consequence of the fact that if I is a left-nilpotent ideal of a left-symmetric algebra A, then the left multiplications L_a are nilpotent for all $a \in I$. and

Proposition (J. Helmstetter (1979). Ann. Inst. Fourier)

Let A be a left symmetric algebra. Then all left multiplications L_a are nilpotent if and only if all the right multiplications R_a are nilpotent and the associated Lie algebra $\mathfrak{g}_A$ is nilpotent.

Similarly, we define an ideal I to be right-nilpotent if there exists some fixed integer $n \geq 1$ such that $R_{a_1} \cdots R_{a_n} = 0$ for all $a_i \in I$. It follows immediately that any right-nilpotent algebra is complete. However, unlike the left-nilpotent case, the largest right-nilpotent ideal need not exist for an arbitrary left-symmetric algebra, because the sum of any two right-nilpotent ideals need not be right-nilpotent. However, we have

Theorem (Zelmanov, E. (1987). Soviet. Math. Dokl.)

A Novikov algebra A has a unique maximal right-nilpotent two-sided ideal $N(A)$, called the right radical of A. Furtheremore, $I(A) = \{a \in A : \operatorname{tr}(R_a) = 0\}$ is a two-sided ideal.

Corollary

Let A be a Novikov algebra over a field $\mathbb{F}$ of characteristic zero. Then, we have $N(A) = R(A) = I(A)$. In particular $R(A)$ is a two-sided ideal.

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Introduction

Special issue in honor of Efim Zelmanov

This issue of Journal of Algebra is dedicated to Efim Zelmanov, one of the most distinguished contemporary mathematicians, who served as an editor for the Journal for eighteen years.

Efim Zelmanov graduated from the Novosibirsk State University in 1977. Until 1990 he worked in Russia, at the Sobolev Institute of Mathematics in Novosibirsk, and since 1991 he works in USA, at the Universities of Wisconsin, Madison, Chicago, Yale, and currently at the University of California, San Diego.

In 1994 he was awarded the Fields Medal for the solution of the Restricted Burnside Problem, a fundamental algebraic conjecture that attracted attention of many famous mathematicians during the XX-th century. He also solved many other long-standing problems in the theories of Lie and Jordan algebras and in group theory.

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Proposition (D. Burde-K. Dekimpe (2006), J. Geom. Phy.)

Any finite-dimensional Lie algebra over k admitting a Novikov structure is solvable.

Proof.

Assume that $\mathfrak{g}$ admits a Novikov structure given by $(x, y) \mapsto x \cdot y$. If $\bar{k}$ denotes the algebraic closure of k, then this Novikov structure extends to a Novikov structure on the Lie algebra $\mathfrak{g} \otimes \bar{k}$ over $\bar{k}$. Therefore, there is no loss in generality in assuming that k itself is algebraically closed. Denote by $R(A)$ the radical of A. Now $R(A)$ is a complete left-symmetric algebra. It is known that the Lie algebra of a complete LSA is solvable, see [D. Segal (1992), Math. Ann.]. Hence the Lie algebra h of $R(A)$ is solvable. On the other hand $A/R(A)$ is a direct sum of fields. It follows that the Lie algebra of $A/R(A)$ is abelian. Hence $\mathfrak{g}/\mathfrak{h}$ is abelian, and $\mathfrak{h}$ is solvable. It follows that $\mathfrak{g}$ is solvable. $\square$

Many Thanks